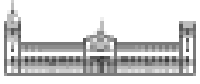

UNIVERSITAT DE BARCELONA

IMUB Institut de Matemàtica

Thanks

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Grant Support: NIH, NSF, ARC (via P. Hall)

Abraham Wald

Three Talks

1. **Function Estimation & Classical Normal Theory**
   - $X_n \sim N_{p(n)}(\theta_n, I)$ $\quad$ $p(n) \nearrow$ with n $\quad (MVN)$

2. **The Threshold Selection Problem**
   - In (MVN) with, say, $\hat{\theta}_i = X_i I\{|X_i| > \hat{t}\}$
   - How to select $\hat{t} = \hat{t}(X)$ "reliably"?

3. **Large Covariance Matrices**
   - $X_n \sim N_{p(n)}(I \otimes \Sigma_{p(n)})$; especially $X_n = \begin{bmatrix} Y_n \\ Z_n \end{bmatrix}$
   - spectral properties of $n^{-1} X_n X_n^T$
   - PCA, CCA, MANOVA

Focus on Gaussian Models

Cue from Wald (1943), *Trans. Amer. Math. Soc.*

1943] TESTS OF STATISTICAL HYPOTHESES 433

4. **Reduction of the general problem to the case of a multivariate normal distribution.** In this section we shall prove two lemmas which will enable us to reduce the general problem of large sample inference to the case where the variates under consideration have a joint normal distribution.

LEMMA 1. *For each positive integer n there exists a set-function $W_n^*(W_n)$*

-an inspiration for Le Cam's theory of Local Asymptotic Normality (LAN)

Focus on Gaussian Models, ctd.

- "Growing models": $p \asymp n$ or $p \gg n$ is now commonplace (classification, genomics..)

- Reality check:

  "But, no real data is Gaussian..."

  Yes (in many fields), and yet,

  consider the power of fable and fairy tale...

1. Function Estimation and Classical Normal Theory

Theme:

Gaussian White Noise model & multiresolution point of view

allow classical parametric theory of $N_d(\theta, I)$

to be exploited in *nonparametric* function estimation

Reference: book manuscript in (non-)progress:

Function Estimation and Gaussian Sequence Models

`www-stat.stanford.edu/~imj`

Agenda

- Classical Parametric Ideas
- Nonparametric Estimation and
  Growing Gaussian Models
- I. Kernel Estimation and James-Stein Shrinkage
- II. Thresholding and Sparsity
- III. Bernstein-von Mises phenomenon

Classical Ideas

a) Multinormal shift model

$X_1, \ldots, X_n$ data from $P_\eta(dx) = f_\eta(x)\mu(dx), \quad \eta \in H \subset \mathbb{R}^d$.

Let $I_0 =$ Fisher information matrix at η_0.

Local asymptotic Gaussian approximation:

$$\{P^n_{\eta_0+\theta/\sqrt{n}}\ ,\ \theta \in \mathbb{R}^d\} \approx \{N_d(\theta, I_0^{-1})\ ,\ \theta \in \mathbb{R}^d\}$$

Classical Ideas, ctd

b) ANOVA, Projections and Model Selection

$$\underset{n\times 1}{Y} = \underset{n\times d}{X}\ \underset{d\times 1}{\beta} + \sigma\epsilon$$

Submodels: $X = [X_0\ X_1] \rightarrow$ projections P_{X_0}.

Canonical form: $y = \theta + \sigma z.$

Projections: $(P_0 y)_i = \begin{cases} y_i & i \in I_0 \\ 0 & \text{o/w} \end{cases}$

MSE: $E\|P_0 y - \theta\|^2 = \sum_{i\in I_0} \sigma^2 + \sum_{i\notin I_0} \theta_i^2$

Classical Ideas, ctd

c) Minimax estimation of θ

$$\inf_{\hat{\theta}} \sup_{\theta \in \mathbb{R}^d} E_\theta \|\hat{\theta}(y) - \theta\|^2 = d\sigma^2$$

attained by MLE $\hat{\theta}_{MLE}(y) = y$.

d) James-Stein estimate

$$\hat{\theta}^{JS}(y) = \left(1 - \frac{d-2}{\|y\|^2}\right) y$$

dominates MLE if $d \geq 3$.

Classical Ideas, ctd

e) Conjugate priors

Prior: $\theta \sim N(0, \tau^2 I)$, Likelihood: $y|\theta \sim N(\theta, \sigma^2 I)$

Posterior: $\theta|y \sim N(\hat{\theta}_y, \sigma_y^2)$

$$\hat{\theta}_y = \frac{\tau^2}{\sigma^2 + \tau^2} y \qquad \sigma_y^2 = \frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2}$$

f) Unbiased risk estimate

$Y \sim N_d(\theta, I)$. For g (weakly) differentiable

$$E\|Y + g(Y) - \theta\|^2 = E[d + 2\nabla^T g(Y) + \|g(Y)\|^2] = E[U(Y)]$$

If $g = g(Y; t)$ then $U = U(Y; t)$.

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Nonparametric function estimation

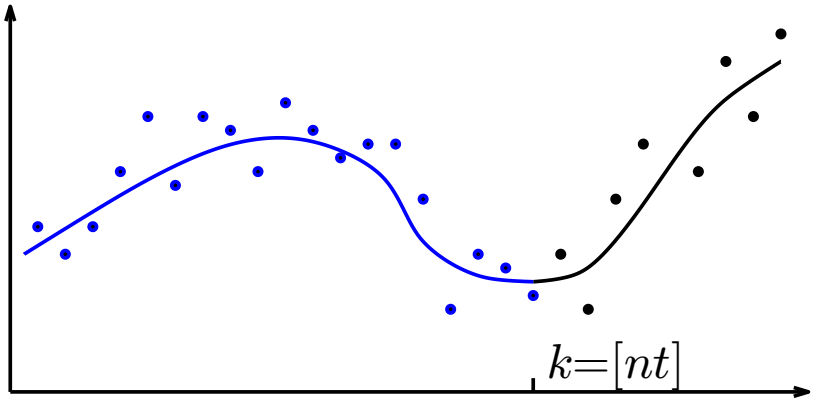
- Hodges and Fix (1951) nonparametric **classification**
- **spectrum estimation** in time series,
- kernel methods for **density est'n, regression**
- roughness penalty methods (splines)
- *techniques largely differ from parametric normal theory*

- Ibragimov & Hasminskii (and school): importance of **Gaussian white noise (GWN) model**:

$$Y_t = \int_0^t f(s)ds + \epsilon W_t, \qquad 0 \le t \le 1$$

Motivation for GWN model

GWN model emerges as large-sample limit of *equispaced regression*, density estimation, spectrum estimation,...



$$y_j = f(j/n) + \sigma w_j$$
$$j = 1, \ldots, n$$

$$\epsilon = \sigma/\sqrt{n}$$

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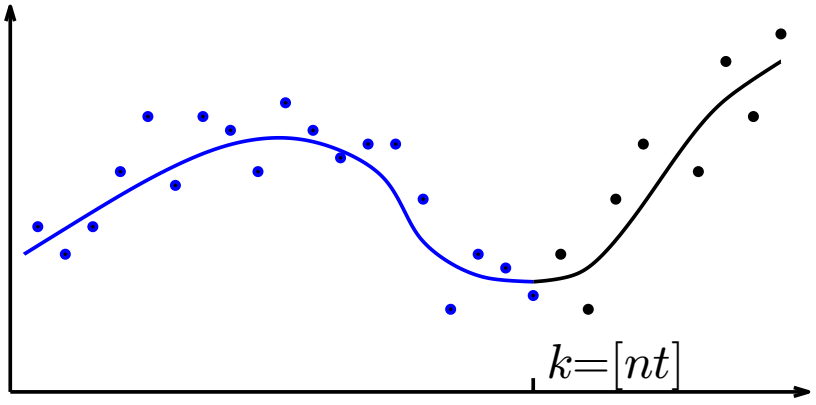

$$y_j = f(j/n) + \sigma w_j$$
$$j = 1, \ldots, n$$

$$\epsilon = \sigma/\sqrt{n}$$

$$\frac{1}{n}\sum_{j=1}^{[nt]} y_j \quad = \quad \frac{1}{n}\sum_{j=1}^{[nt]} f(j/n) + \quad \frac{\sigma}{\sqrt{n}}\frac{1}{\sqrt{n}}\sum_{j=1}^{[nt]} w_j$$

Motivation for GWN model

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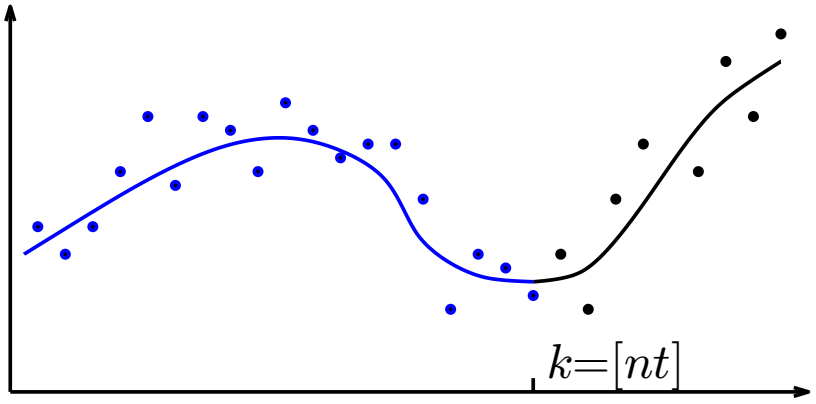

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$$\frac{1}{n}\sum_{j=1}^{[nt]} y_j \quad = \quad \frac{1}{n}\sum_{j=1}^{[nt]} f(j/n) + \quad \frac{\sigma}{\sqrt{n}}\frac{1}{\sqrt{n}}\sum_{j=1}^{[nt]} w_j$$

$$Y_t \quad = \quad \int_0^t f(s)ds \quad + \quad \frac{\sigma}{\sqrt{n}} \quad \cdot \quad W_t$$

Series form of WN Model

$$Y_t = \int_0^t f(s)ds + \epsilon W_t$$

For *any* orthonormal basis $\{\psi_\lambda, \lambda \in \Lambda\}$ for $L_2[0,1]$,

$$\int \psi_\lambda dY_t = \int \psi_\lambda f dt + \epsilon \int \psi_\lambda dW_t$$

$$\Rightarrow \qquad \boxed{y_\lambda = \theta_\lambda + \epsilon z_\lambda, \qquad (z_\lambda) \overset{i.i.d.}{\sim} N(0,1), \quad \lambda \in \Lambda}$$

Parseval relation implies

$$\int (\hat{f} - f)^2 = \sum (\hat{\theta}_\lambda - \theta_\lambda)^2 = \|\hat{\theta} - \theta\|^2, \qquad \text{etc.}$$

$\longrightarrow$ analysis of **infinite** sequences in $\ell_2(\mathbb{N})$

Multiresolution connection

Wavelet orthonormal bases $\{\psi_{jk}\}$ have **double** index

level ("octave")	$j = 1, 2, \ldots,$
location	$k = 1, \ldots, 2^j$

Collect coefficients in a single level:

$$y_j = (y_{jk}) \qquad k = 1, \ldots, 2^j$$

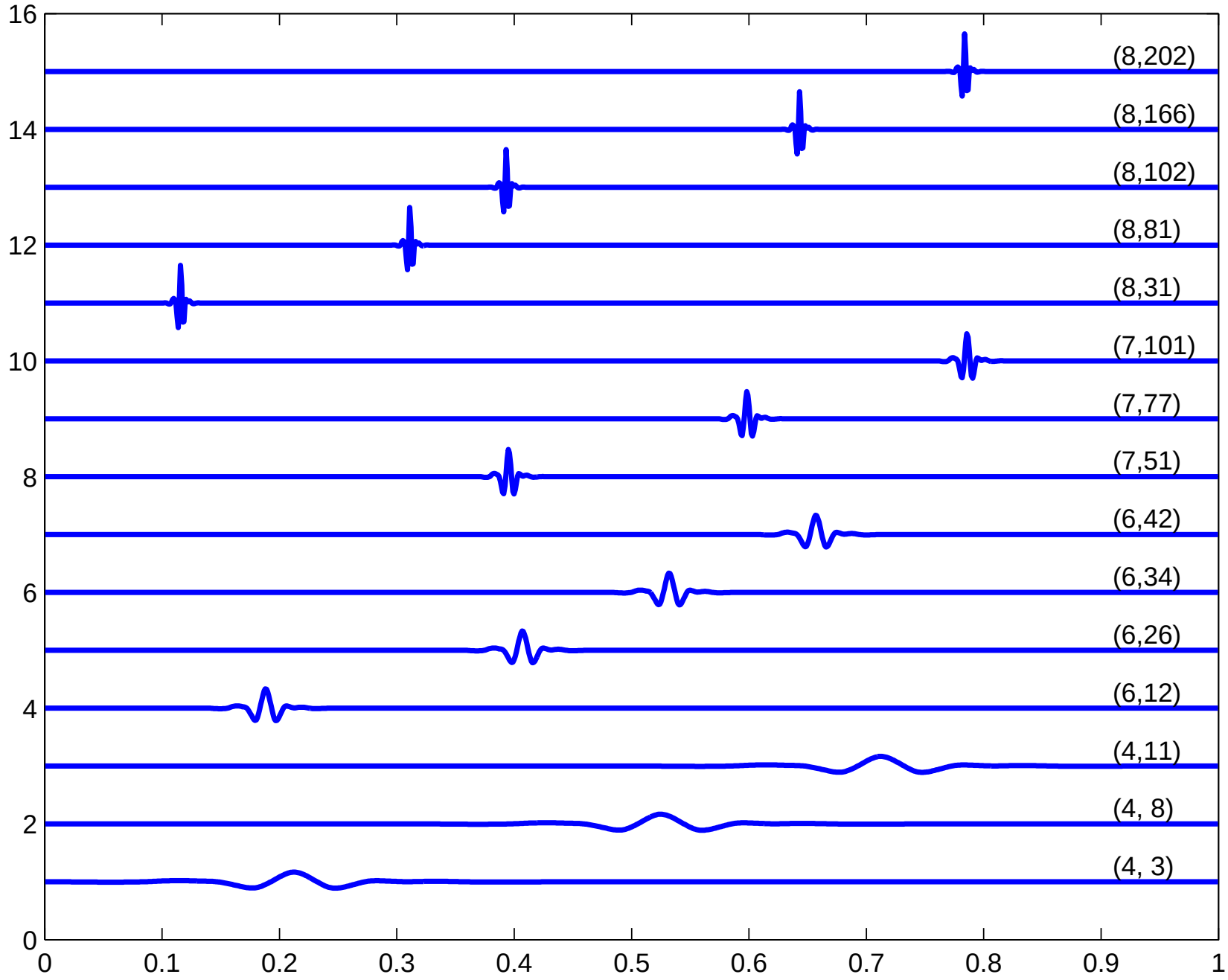
$$\theta_j = (\theta_{jk}) \qquad \text{etc.}$$

Finite (but growing with j) multivariate normal model

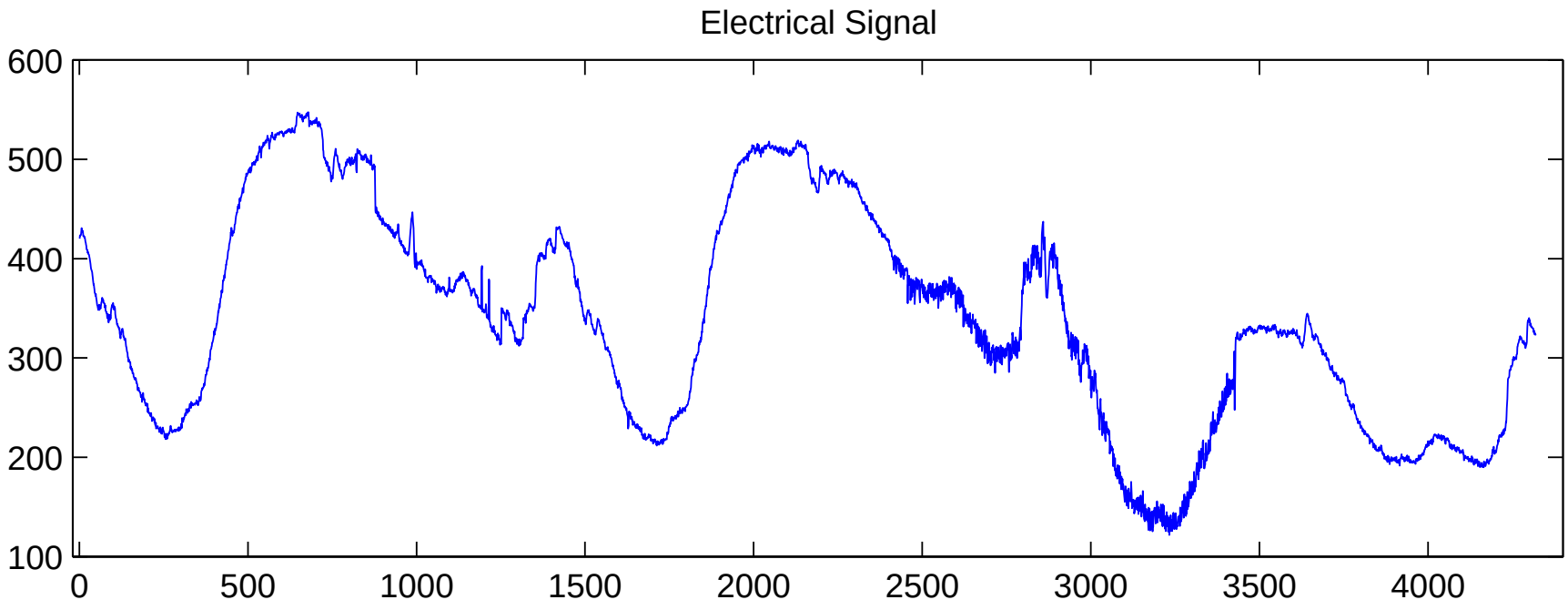
$$y_j \sim N_{2^j}(\theta_j, \epsilon^2 I), \qquad j = 1, 2, \ldots$$

$\Rightarrow$ apply classical normal theory to the vectors y_j.

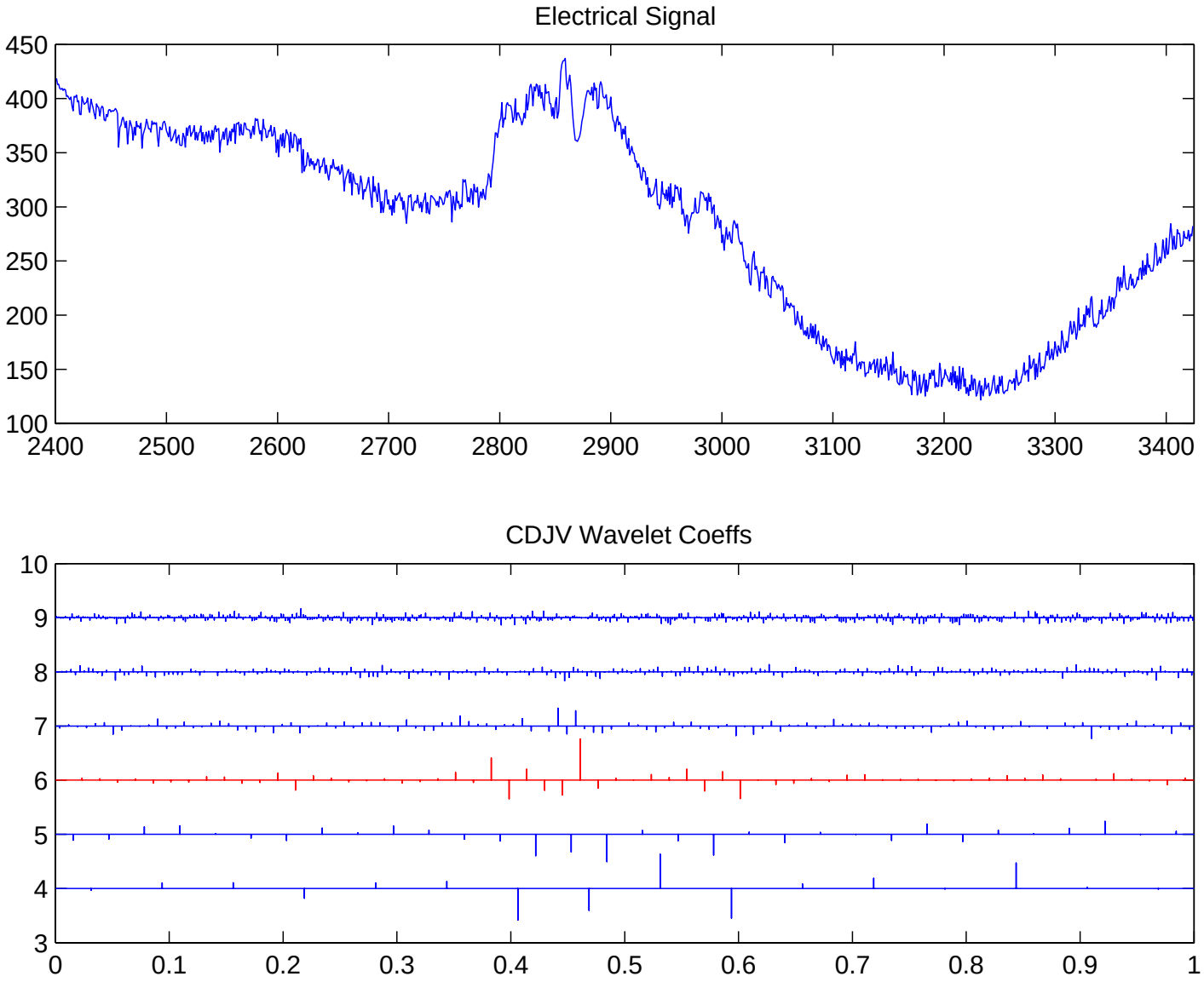
Some S8 Symmlets at various scales and locations



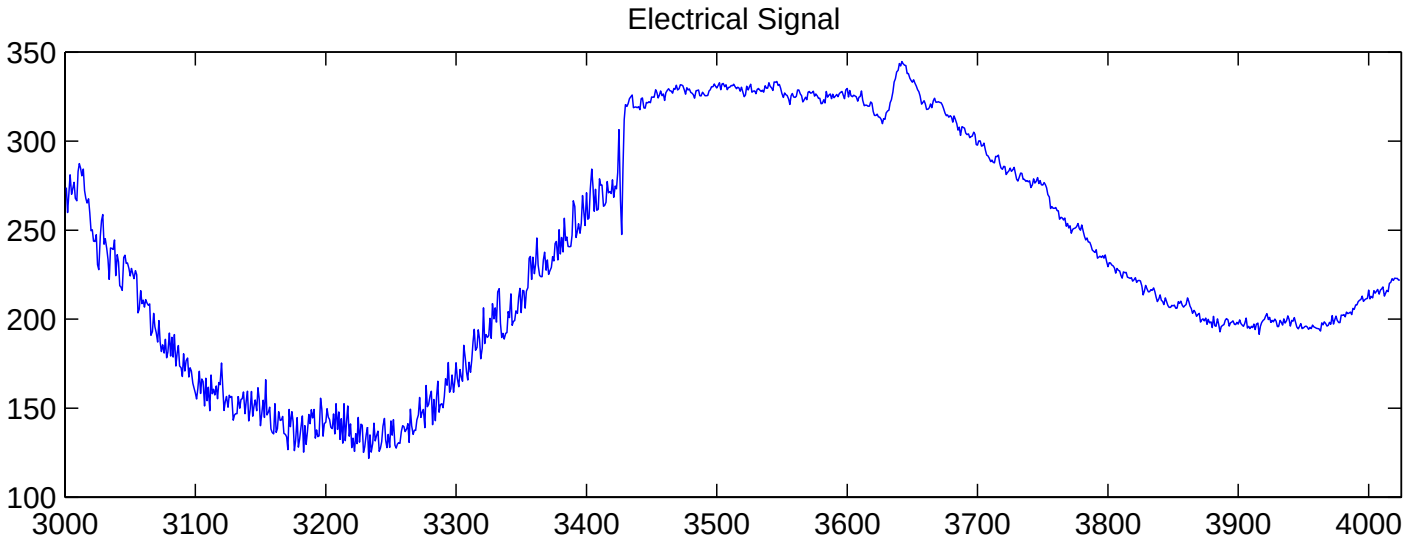
Example

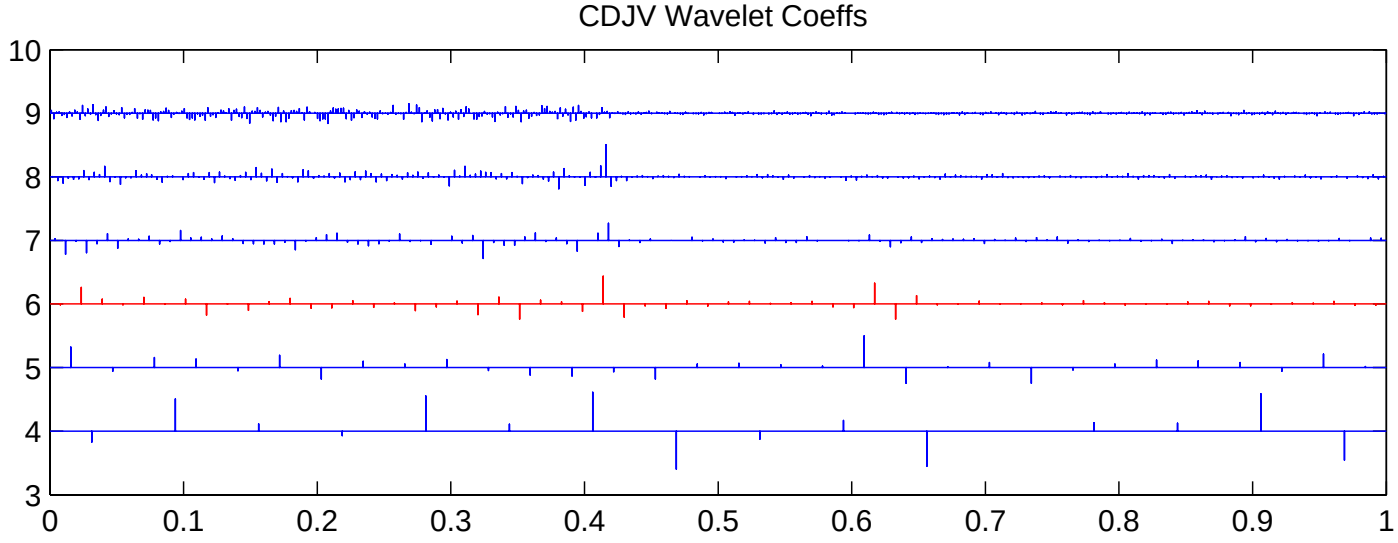


- Nationwide electricity consumption in France
- Here: 3 days (Thur, Fri, Sat), summer 1990, $\Delta t = 1$min.
- from Misiti et. al., *Rev. Statist. Appliquée* **1994**, 57-77.
- N.B. Malfunction of meters: d. 2, 16.30 – d. 3, 09.00



Often o.k. to treat a single level as $\sim N_{2^j}(\theta_j, \epsilon^2 I)$





Or, apply $N_d(\theta, I)$ model to a block **within** a level.

The Minimax principle

Given *loss function* $L(a, \theta)$, and *risk*

$$R(\hat{\theta}, \theta) = E_\theta L(\hat{\theta}, \theta)$$

choose estimator $\hat{\theta}$ so as to minimize the *maximum risk*

$$\sup_{\theta \in \Theta} E_\theta L(\hat{\theta}, \theta).$$

- introduced into statistics by Wald
- L.J. Savage (1951):

  "the only rule of comparable generality proposed since [that of] Bayes' was published in 1763"

- standard complaint: $\hat{\theta}$ depends on Θ:
  - optimizing on the worst case may be irrelevant

Simultaneous near-minimaxity

Shift in perspective on minimaxity

- fix $\hat{f}$ in advance; an estimator to be evaluated
- for many spaces $\mathcal{F}$, compare

$$\sup_{\mathcal{F}} R(\hat{f}, f) \quad \text{to} \quad \mathcal{R}(\mathcal{F}) = \inf_{\hat{f}} \sup_{\mathcal{F}} R(\hat{f}, f)$$

- shift *from*

  "Exact answer to 'wrong' problem"

  to

  "Approx answer to many (related) problems"

$\Rightarrow$ an imperfect, yet serviceable tool

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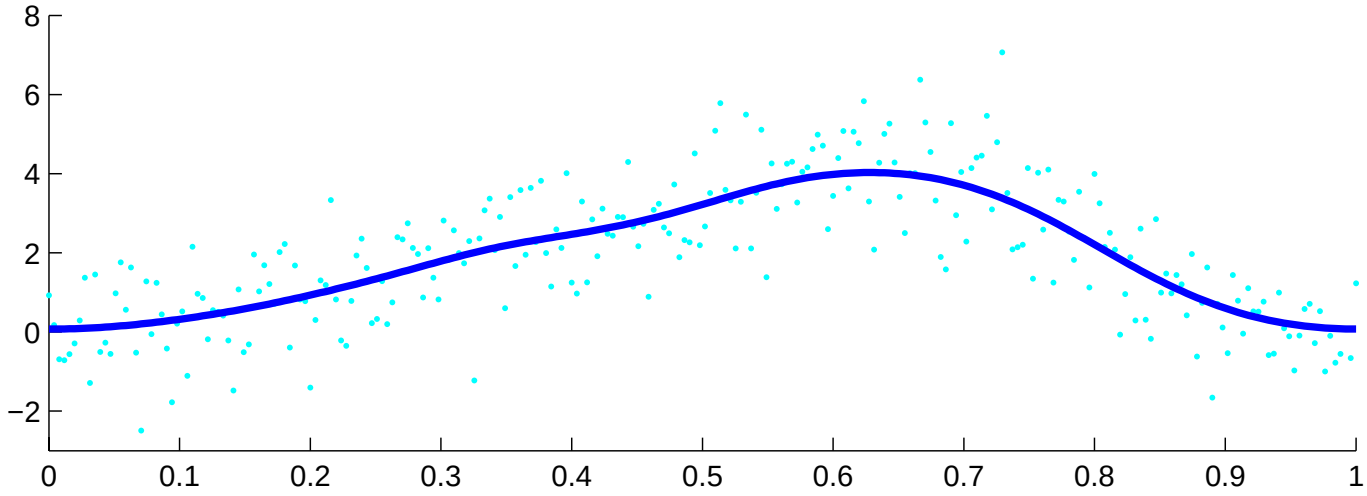
I. Kernel estimation & James-Stein Shrinkage



Priestley-Chao estimator:

$$\hat{f}_h(t) = \frac{1}{Nh} \sum_{i=1}^{N} K\Big(\frac{t - t_i}{h}\Big) Y_i \qquad \text{e.g. } K(x) = (1 - x^2)_+^4$$

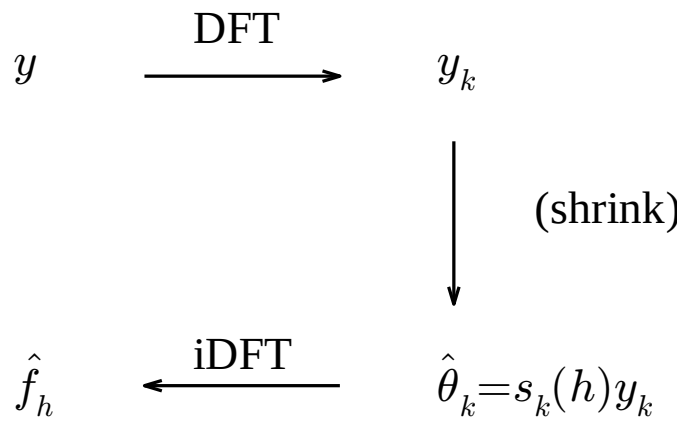
- Automatic choice of h?
- Huge literature, (e.g. Wand & Jones, 1995)
- James-Stein provides simple, powerful approach

Fourier Form of Kernel Smoothing

Convolution $\longrightarrow$ multiplication

Shrinkage factors

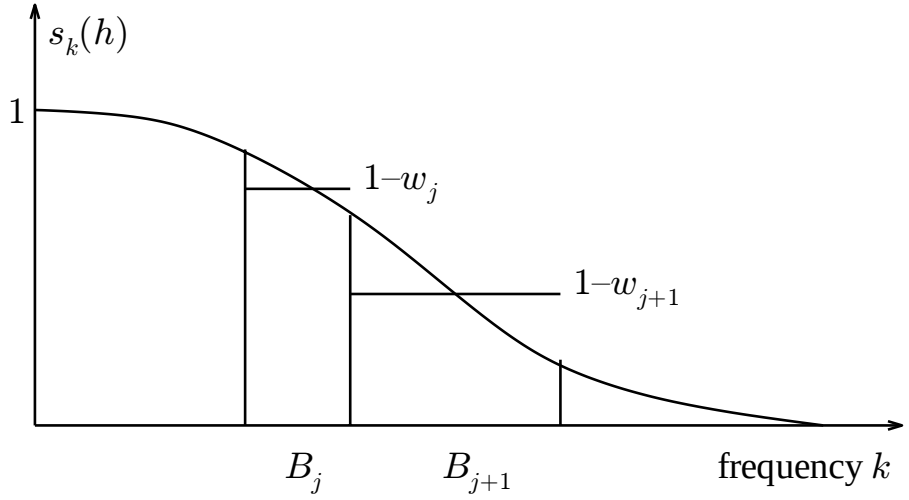
- $s_k(h) \in [0, 1]$
- decrease with frequency



Flatten shrinkage in blocks:

$$\hat{\theta}_k = (1 - w_j) y_k \qquad k \in B_j$$

- How to choose w_j?



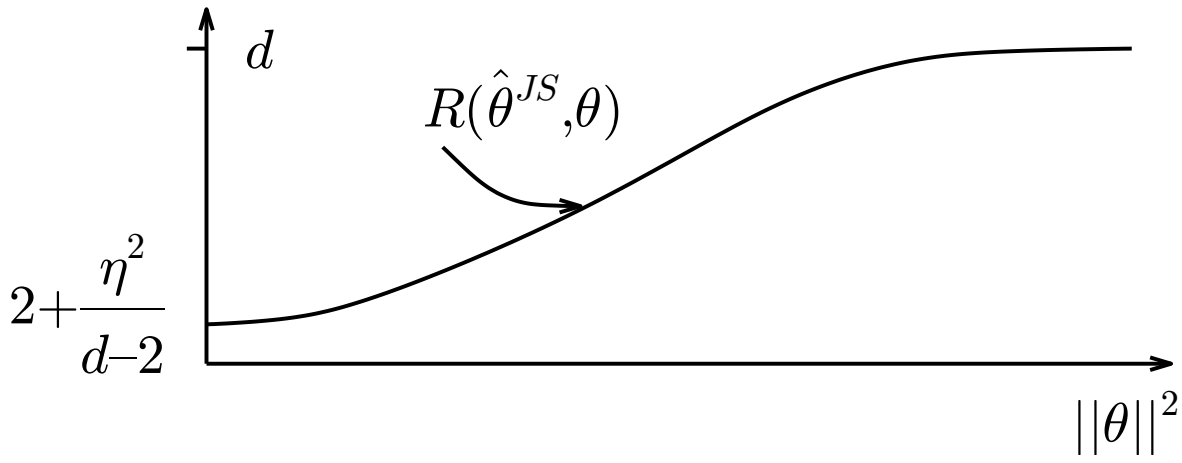
James-Stein Shrinkage

For $y \sim N_d(\theta, I)$, and $\mathbf{d \geq 3}$

$$\hat{\theta}^{JS+} = \left(1 - \frac{d-2+\eta}{|y|^2}\right)_+ y$$

Unbiased estimate of risk:

$$E\|\hat{\theta}^{JS} - \theta\|^2 = E_\theta\left\{d - \frac{(d-2)^2+\eta^2}{\|y\|^2}\right\} \leq d$$



Smoothing via Blockwise James-Stein

Block Fourier or wavelet (here) coefficients:

$$y_j = (y_{jk}), \quad k = 1, \ldots, d_j = 2^j; \qquad \theta_j = (\theta_{jk}) \quad \text{etc.}$$

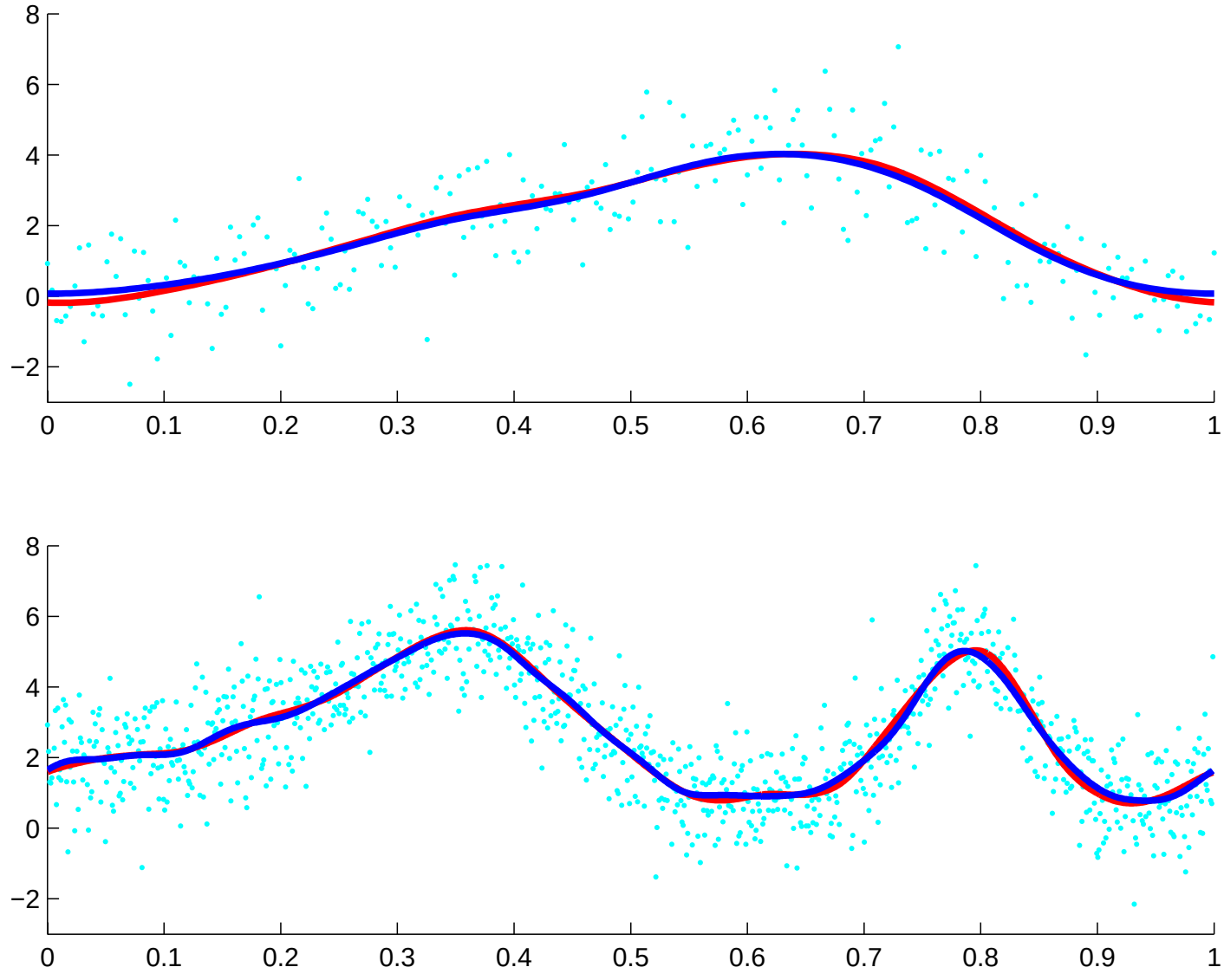
Apply J-S shrinkage to each block: **[refs]**

$$\hat{\theta}_j^{JS+} = \left(1 - \frac{\beta_j \epsilon^2}{|y_j|^2}\right)_+ y_j \qquad 2 \le j \le \log_2 N$$

$$\beta_j = \begin{cases} d_j - 2 & (\texttt{ExactJS}) \textbf{ or} \\ d_j + \sqrt{2d_j}\sqrt{2\log d_j} & (\texttt{ConsJS}) \end{cases}$$

For `ConsJS`, if $\theta_j = 0$, then $P\{|y_j|^2 < \beta_j\} \nearrow 1$, so $\hat{\theta}_j^{JS} = 0$.

Block James-Stein imitates Kernel



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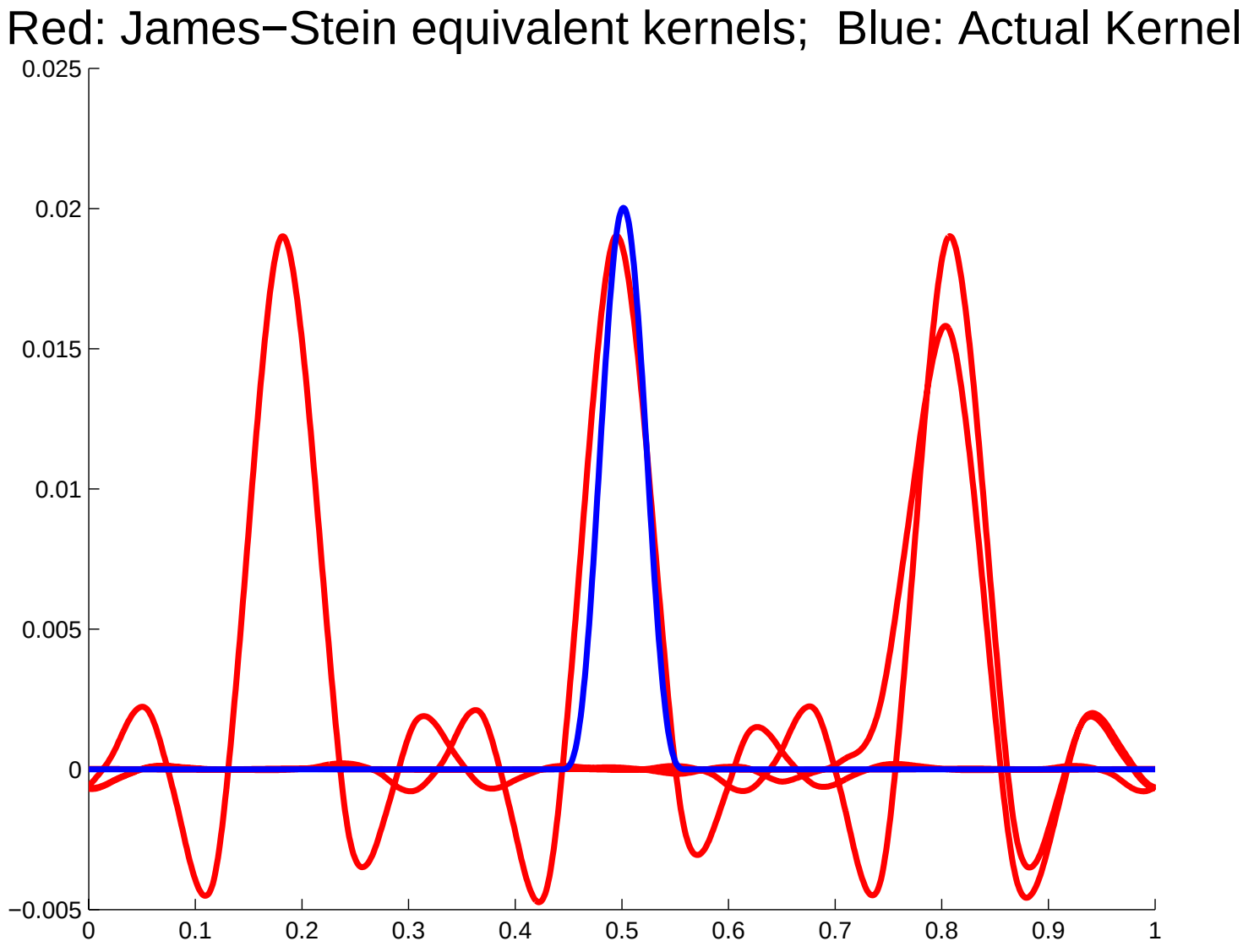

Properties of Block James-Stein

- Block JS imitates kernel smoothing, **but**
- near automatic choice of 'bandwidth'
  – canonical choices of β_j
- Easy theoretical analysis from unbiased risk bounds

$$E\|\hat{\theta}_j^{JS} - \theta_j\|^2 \leq 2\epsilon^2 + \|\theta_j\|^2 \wedge 2^j\epsilon^2$$

Properties of Block James-Stein

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$$E\|\hat{\theta}_j^{JS} - \theta_j\|^2 \leq 2\epsilon^2 + \|\theta_j\|^2 \wedge 2^j\epsilon^2$$

- e.g. below: MSE for *Hölder smooth* functions:
  - $0 < \delta < 1 \quad |f(x) - f(y)| \leq B|x-y|^\delta$ all x, y (*)
  - $\alpha = r + \delta, r \in \mathbb{N}, \qquad D^r f$ satisfies (*).
- In terms of wavelet coefficients:

$$f \in \mathcal{H}_\alpha(C) \quad \Leftrightarrow \quad |\theta_{jk}| \leq C2^{-(\alpha+1/2)j} \quad \text{for all } j, k$$

A single level determines MSE

From unbiased risk bounds and Hölder smoothness:

$$E\|\hat{\theta}_j^{JS} - \theta_j\|^2 \leq \qquad [2\epsilon^2 + \|\theta_j\|^2 \wedge 2^j\epsilon^2]$$

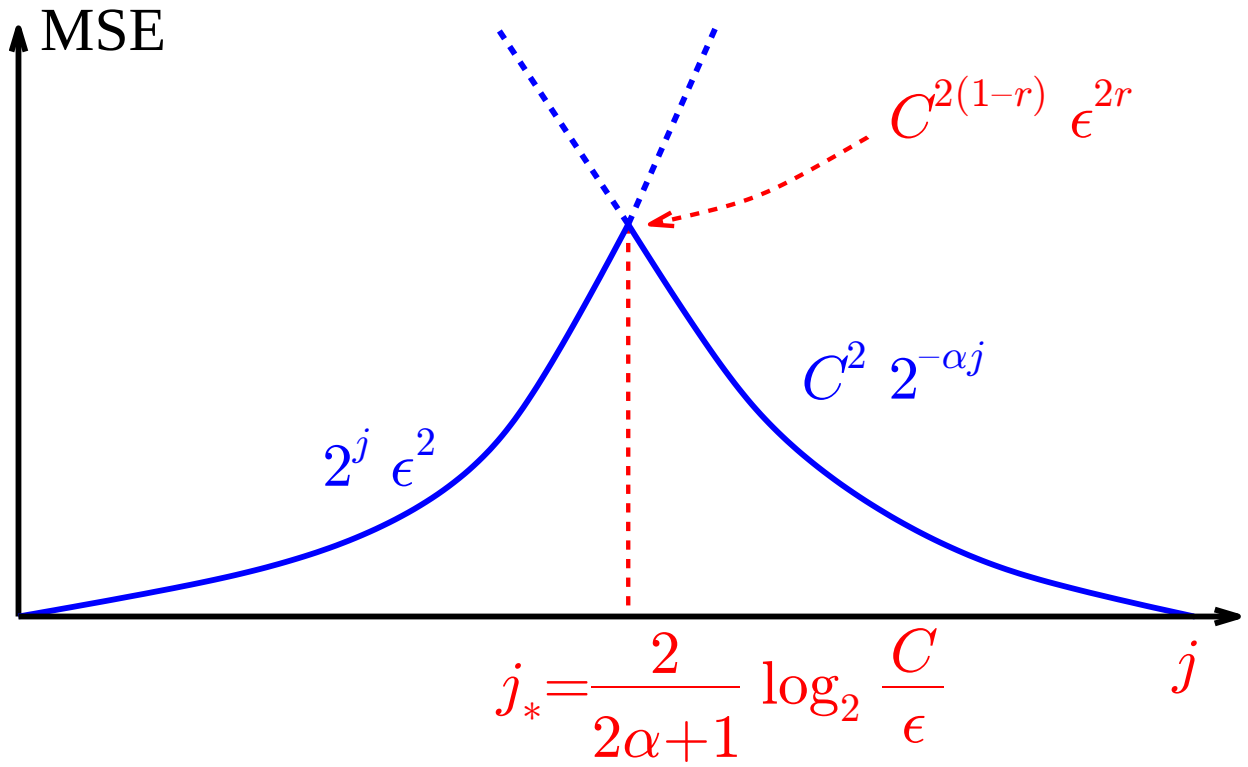
$$f \in \mathcal{H}_\alpha(C) \Rightarrow \|\theta_j\|^2 \leq C^2 2^{-2\alpha j}$$

A single level determines MSE

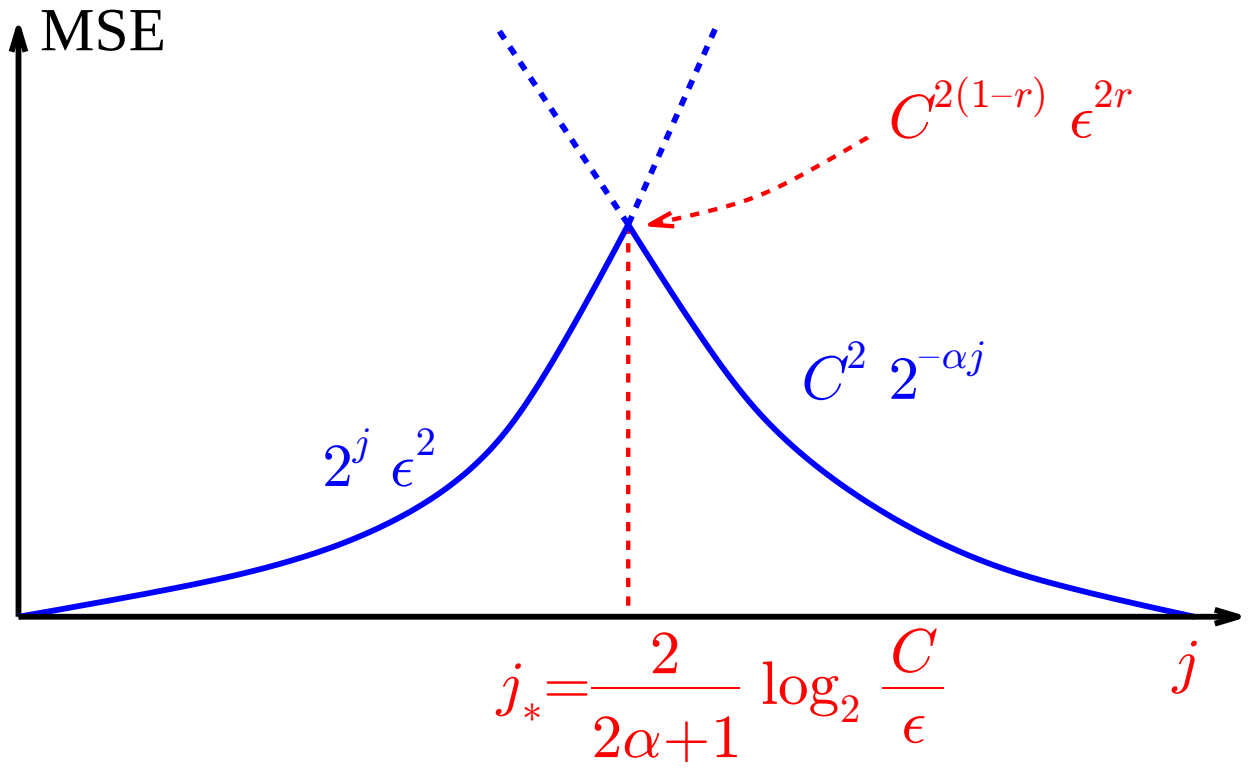
From unbiased risk bounds and Hölder smoothness:

$$\sum_j E\|\hat{\theta}_j^{JS} - \theta_j\|^2 \leq \sum_{j \leq J} [2\epsilon^2 + \|\theta_j\|^2 \wedge 2^j \epsilon^2] + \sum_{j > J} \|\theta_j\|^2$$

$$f \in \mathcal{H}_\alpha(C) \Rightarrow \|\theta_j\|^2 \leq C^2 2^{-2\alpha j}$$



Growing Gaussians



- geometric decay of MSE away from critical j_*
- **Growing Gaussian** aspect: As noise $\epsilon = \sigma/\sqrt{n}$) *decreases*, worst level $j_* = j_*(\epsilon, C)$ *increases:*

$$d_{j_\star} = 2^{j_*} = c(C/\epsilon)^{2/(2\alpha+1)}$$

MultiJS is rate-adaptive

The "optimal rate" for $\mathcal{H}_\alpha(C)$ is ϵ^{2r}, with $r = r(\alpha) = \frac{2\alpha}{2\alpha+1}$.

- JS risk bounds imply simultaneous near-minimaxity:

$$\sup_{\mathcal{H}_\alpha(C)} R(\hat{f}^{JS}, f) \leq cC^{2(1-r)}\epsilon^{2r}(1+o(1)) \asymp \mathcal{R}(\mathcal{H}_\alpha(C)).$$

- For a **single, prespecified** estimator $\hat{f}^{JS}$, valid for
  - all smoothness $\alpha \in (0, \infty)$,
  - bounds $C \in (0, \infty)$.
- No "speed limit" to rate of convergence
  - "infinite order kernel" (for certain wavelets)
- Conclusion follows easily from single level James-Stein analysis in multinormal mean model

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James-Stein Fails on Sparse Signals

$$\frac{d\|\theta\|^2}{d+\|\theta\|^2} \leq R(\hat{\theta}^{JS}, \theta) \leq 2 + \frac{d\|\theta\|^2}{d+\|\theta\|^2}$$

$r-$**spike** θ_r**:** r co-ords at $\sqrt{d/r}$ $\quad \Rightarrow \quad \|\theta\|^2 = d.$

$$. | \, . \, . | \, . \, . \, . | \, . \, .$$

So $\qquad R(\hat{\theta}^{JS}, \theta_r) \geq d/2$

James-Stein Fails on Sparse Signals

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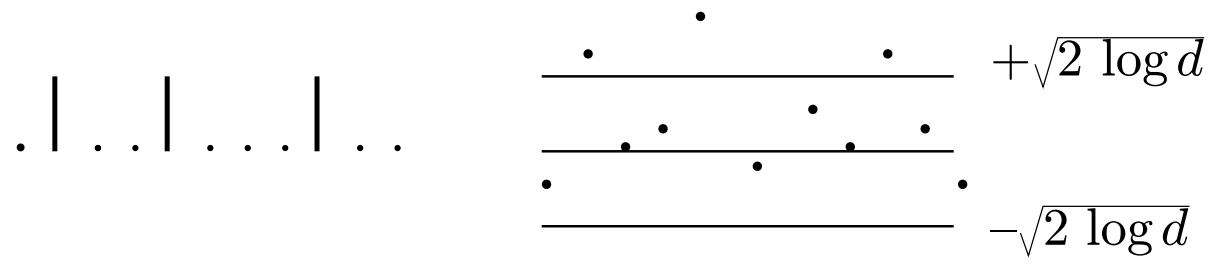


So $\qquad R(\hat{\theta}^{JS},\theta_r) \ge d/2$

Hard thresholding at $t = \sqrt{2\log d}$:

$$R(\hat{\theta}^{HT},\theta_r) \approx d\cdot 2t\phi(t) + r\cdot[1+\eta(d/r)]$$
$$\le r + 3\sqrt{2\log d}$$

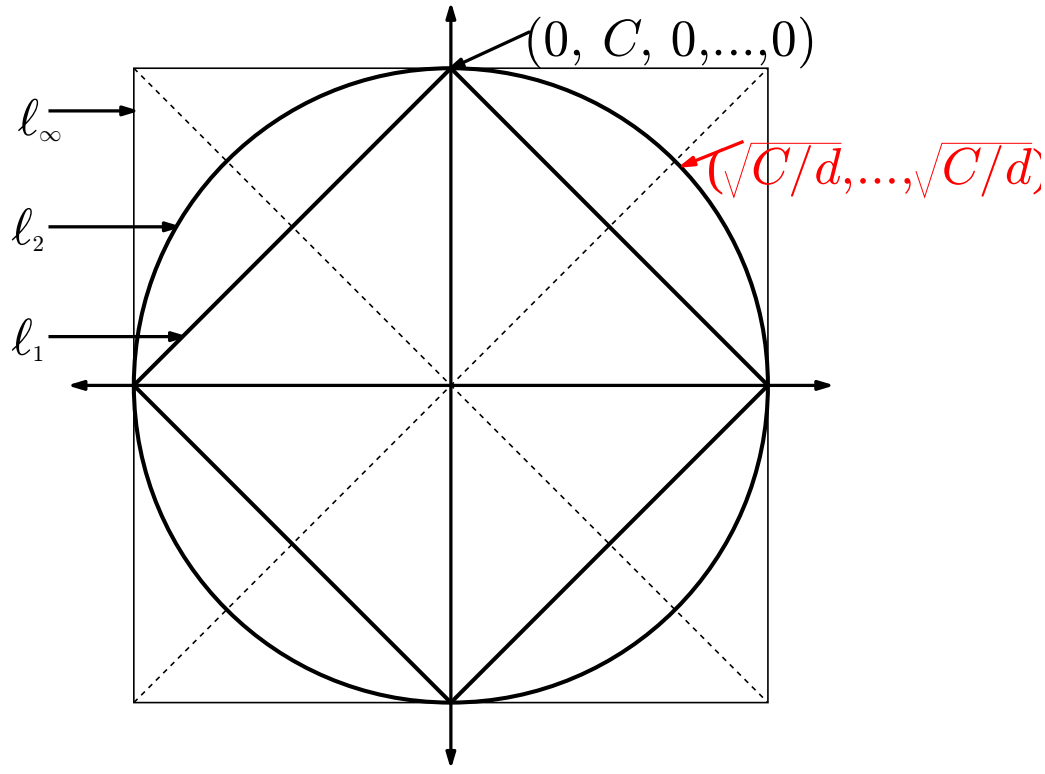
A more systematic story for thresholding

Based on ℓ_p norms and balls: (mostly with $p < 2$)

$$\|\theta\|_p^p = \sum_{i=1}^{d} |\theta_i|^p$$

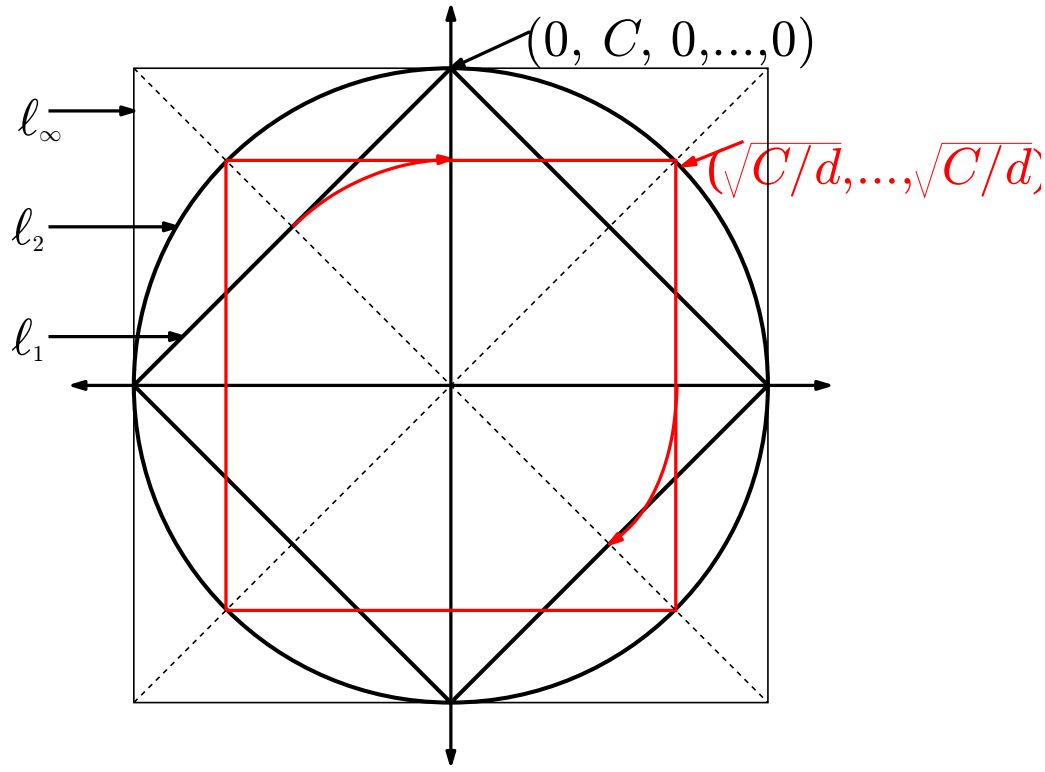
1. ℓ_p norms capture sparsity
2. thresholding arises from least squares estimation with ℓ_p constraints
3. describe best possible estimation over ℓ_p balls
4. show that thresholding (nearly) attains the best possible

ℓ_p norms, $p < 2$, capture sparsity



ℓ_1 ball smaller than ℓ_2, $\Rightarrow$ expect better estimation

ℓ_p norms, $p < 2$, capture sparsity



ℓ_1 ball smaller than ℓ_2, $\Rightarrow$ expect better estimation

Sparsity of representation depends on basis: rotation sends

$$(0, C, 0, \ldots) \quad \rightarrow \quad (C/\sqrt{d}, \ldots, C/\sqrt{d})$$

Thresholding from constrained least squares

$$\min \sum (y_i - \theta_i)^2 \quad \text{s.t.} \quad \sum |\theta_i|^p \le C^p$$

i.e. $$\min \sum (y_i - \theta_i)^2 + \lambda |\theta_i|^p$$

leads to

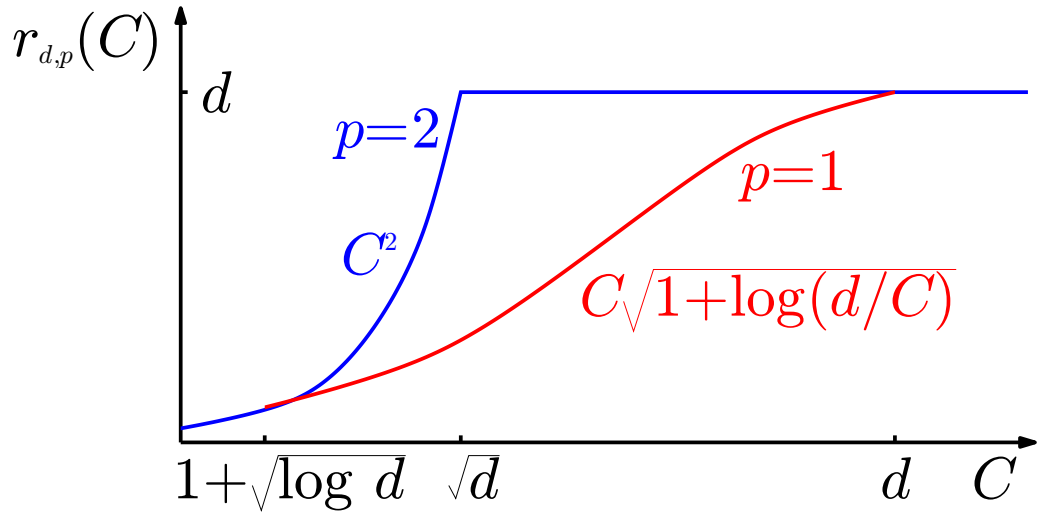
- $p = 2$ **Linear Shrinkage** $\hat{\theta}_i = (1+\lambda)^{-1} y_i$
- $p = 1$ **Soft thresholding** $$\hat{\theta}_i = \begin{cases} y_i - \lambda' & y_i > \lambda' \\ 0 & |y_i| \le \lambda' \\ y_i + \lambda' & y_i < -\lambda' \end{cases}$$
  $[\lambda' = \lambda/2]$
- $p = 0$ **Hard thresholding** $\hat{\theta}_i = y_i I\{|y_i| > \lambda\}.$
  [penalty $\lambda \sum I\{\theta_i \neq 0\}$]

Bounds on best estimation on ℓ_p balls

Minimax risk: $R_{d,p}(C) = \inf_{\hat{\theta}} \sup_{\|\theta\|_p \le C} E_\theta \sum_1^d (\hat{\theta}_i - \theta_i)^2.$

Non-asymptotic bounds: [Birgé-Massart]

$$c_1 r_{d,p}(C) \le R_{d,p}(C) \le c_2[\log d + r_{d,p}(C)]$$



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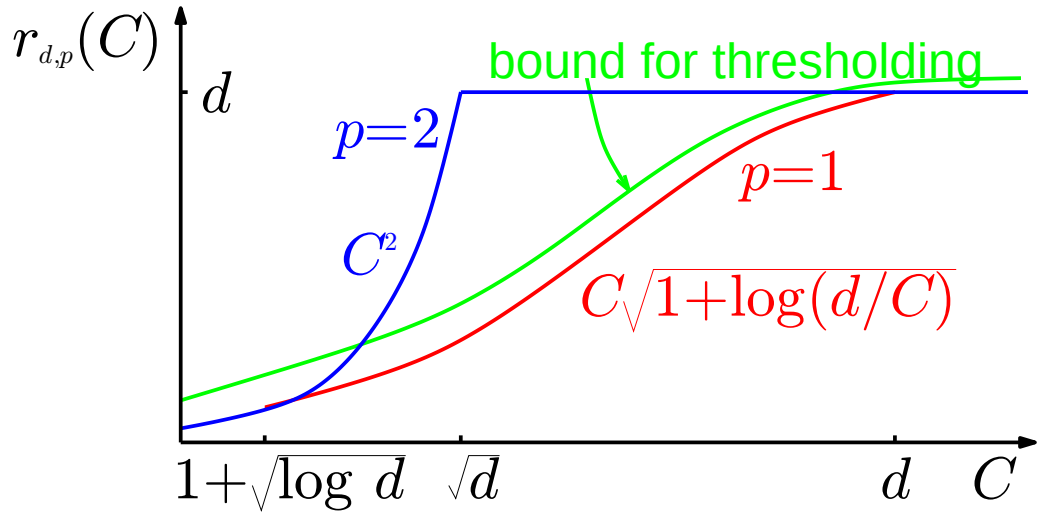

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Thresholding nearly attains the bound

For simplicity: threshold $t = \sqrt{2 \log d}$

Oracle inequality for soft thresholding: [*non-asymptotic!*]

$$E\|\hat{\theta}^{ST} - \theta\|^2 \leq (2 \log d + 1)[1 + \sum(\theta_i^2 \wedge 1)]$$

Apply to ℓ_1 ball $\{\theta : \|\theta\|_1 \leq C\}$.

$$\sup_{\|\theta\|_1 \leq C} E\|\hat{\theta}^{ST} - \theta\|^2 \leq (2 \log d + 1)(C + 1).$$

Better bounds possible for data-dependent thresholds $t = t(Y)$

(Lec. 2)

Summary for $N_d(\theta, I)$

For $X \sim N_d(\theta, I)$, comparing

- James Stein shrinkage ($\beta = d - 2$), and
- soft thresholding at $t = \sqrt{2 \log d}$:

James-Stein shrinkage: orthogonally invariant:

$$\tfrac{1}{2}(\|\theta\|^2 \wedge d) \leq E\|\hat{\theta}^{JS} - \theta\|^2 \leq 2 + (\|\theta\|^2 \wedge d)$$

Thresholding: co-ordinatewise, and co-ordinate dependent:

$$\tfrac{1}{2}\sum(\theta_i^2 \wedge 1) \leq E\|\hat{\theta}^{ST} - \theta\|^2 \leq (2 \log d + 1)(1 + \sum(\theta_i^2 \wedge 1))$$

Implications for Function Estimation

Key example: functions of bounded **total variation:**
$TV(C) = \{f \ : \ \|f\|_{TV} \leq C\}$

$$\|f\|_{TV} = \sup_{t_1 < \ldots < t_N} \sum_{i=1}^{N-1} |f(t_{i+1}) - f(t_i)| + \|f\|_1$$

Well captured by weighted combinations of ℓ_1 **norms on wavelet coefficients:**

$$c_1 \sup_j 2^{j/2} \|\theta_j\|_1 \leq \|f\|_{TV} \leq c_2 \sum_j 2^{j/2} \|\theta_j\|_1$$

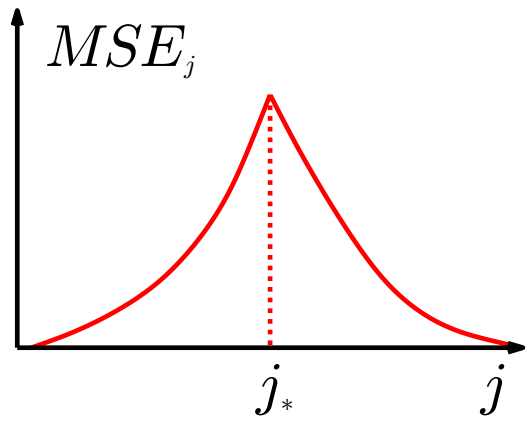
Best possible (minimax) MSE

$$R(TV(C), \epsilon) = \inf_{\hat{f}} \sup_{f \in TV(C)} E\|\hat{f} - f\|^2$$

Reduction to single levels

$$\sup_{\theta} E\|\hat{\theta} - \theta\|^2 = \sum_j \sup_{\theta_j} E\|\hat{\theta}_j - \theta_j\|^2$$

- Apply *thresholding* bounds for each j: to $N_{2^j}(\theta_j, \epsilon^2 I)$.
- $\exists$ a worst case level $j_* = j_*(\epsilon, C)$,
- Geometric decay of $\sup_j E\|\hat{\theta}_j - \theta_j\|^2$ as $|j - j_*| \nearrow$.



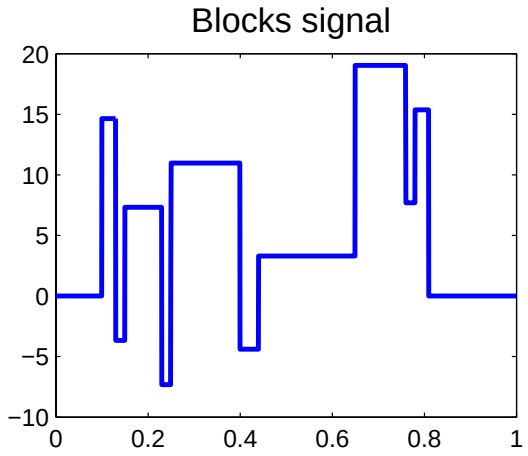
Growing Gaussian aspect: As noise $\epsilon = \sigma/\sqrt{n})$ *decreases*, worst level $j_* = j_*(\epsilon, C)$ *increases:*

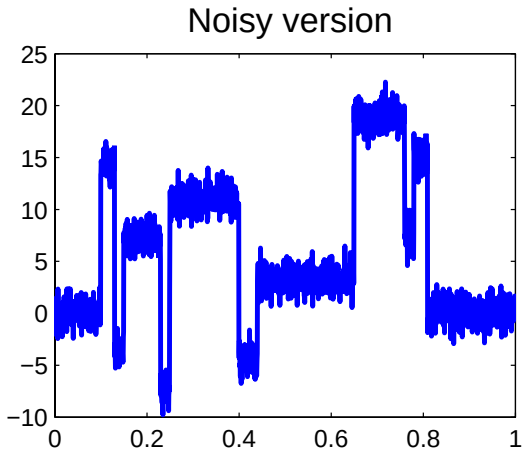
$$d_{j_\star} = 2^{j_*} = c(C/\epsilon)^{2/3}$$

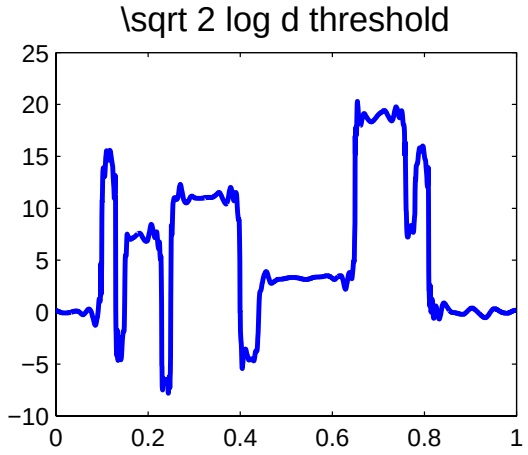
Final Result

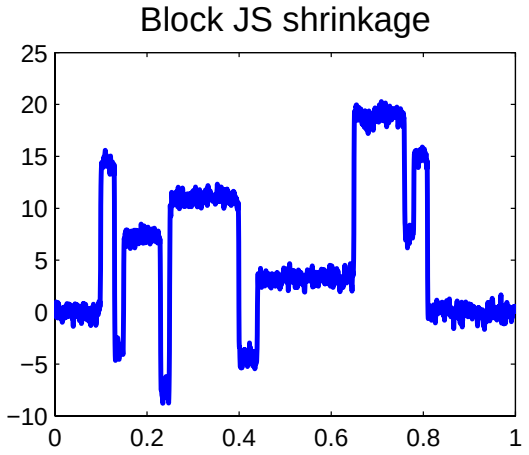
$$\sup_{TV(C)} R(\hat{f}_{thr}, f) \asymp C^{2(1-r)}\epsilon^{2r} \qquad r = 2/3$$

$$\sup_{TV(C)} R(\hat{f}_{JS}, f) \asymp C^{2(1-r_L)}\epsilon^{2r_L} \qquad r_L = 1/2$$









Agenda

- Classical Parametric Ideas
- Nonparametric Estimation and
  Growing Gaussian Models
- I. Kernel Estimation and James-Stein Shrinkage
- II. Thresholding and Sparsity
- III. Bernstein-von Mises phenomenon

Bernstein-von Mises Phenomenon

Asymptotic match of freq. & Bayesian confidence intervals:

Classical version

$$Y_1, \ldots, Y_n \overset{i.i.d.}{\sim} p_\theta(y)d\mu \qquad \theta \in \Theta \subset \mathbb{R}^d \qquad \mathbf{d \text{ fixed}}$$

Under mild conditions, as $n \to \infty$,

$$\left\| P_{\theta|Y} - N\left(\hat{\theta}_{MLE} \ , \ n^{-1} I_{\theta_0}^{-1}\right) \right\| \quad \overset{P_{n,\theta_0}}{\longrightarrow} \quad 0,$$

where

- $I_\theta = E_\theta\left[\frac{\partial}{\partial\theta} \log p_\theta\right]\left[\frac{\partial}{\partial\theta} \log p_\theta\right]^T$ is Fisher information,
- $\|P - Q\| = \max_A |P(A) - Q(A)|$ is total variation distance

Non-parametric Regression

$$Y_i = f(i/n) + \sigma_0 w_i, \qquad i = 1, \ldots, n$$

For typical smoothness priors (e.g. d^{th} integrated Wiener process prior)

- Bernstein - von Mises fails [Dennis Cox, D. A. Freedman]
- frequentist & posterior laws of $\|\hat{f} - f\|_2^2$ **mismatch** in center and scale

Here,

- revisit via elementary Growing Gaussian Model approach
- Apply to single levels of wavelet decomposition

Growing Gaussian Model

Data: $Y_1, \ldots, Y_n | \theta \overset{i.i.d.}{\sim} N_p(\theta, \sigma_0^2 I)$ **Prior:** $\theta \sim N_p(0, \tau_n^2)$.

- growing: $p = p(n) \nearrow$ with n
- $\sigma_n^2 = \text{Var}(\bar{Y}_n|\theta) = \sigma_0^2/n$; τ_n^2 may depend on n.

Goal: compare $\mathcal{L}(\hat{\theta}_{MLE}|\theta), \mathcal{L}(\hat{\theta}_{Bayes}|\theta)$ with $\mathcal{L}(\theta|Y)$.

Posterior: All Gaussian: **centering** and **scaling** are key:

$$\boxed{\mathcal{L}\big(\theta|\bar{Y} = \bar{y}\big) \sim N_p\big(\ \hat{\theta}_B = \mathbf{w_n}\bar{y},\ \mathbf{w_n}\sigma_n^2 I\ \big)}$$

$$\mathbf{w_n} = \frac{\tau_n^2}{\sigma_n^2 + \tau_n^2}$$

Growing Gaussians and BvM

Correspondences:

$$\begin{array}{rcl} P_{\theta|Y} & \leftrightarrow & N_p(w_n\bar{y},\ w_n\sigma_n^2 I\ ) \\ N\big(\hat{\theta}_{MLE}, n^{-1}I_{\theta_0}^{-1}\big) & \leftrightarrow & N_p(\quad \bar{y},\quad \sigma_n^2 I\ ) \end{array}$$

For BvM to hold, now need $w_n \nearrow 1$ sufficiently fast:

Proposition: In the growing Gaussian model

$$\big\| P_{\theta|Y} - N\big(\hat{\theta}_{MLE}, n^{-1}I_{\theta_0}^{-1}\big)\big\| \quad \stackrel{P_{n,\theta_0}}{\longrightarrow} \quad 0,$$

if and only if

$$\sqrt{p}\frac{\sigma_n^2}{\tau_n^2} = \frac{\sqrt{p}}{n}\frac{\sigma_0^2}{\tau_n^2} \to 0 \qquad \text{i.e.} \quad w_n = 1 - o\Big(\frac{1}{\sqrt{p}_n}\Big).$$

Example: Pinsker priors

Back to regression: $dY_t = f(t)dt + \sigma_n dW_t$

Minimax MSE linear estimation of f:

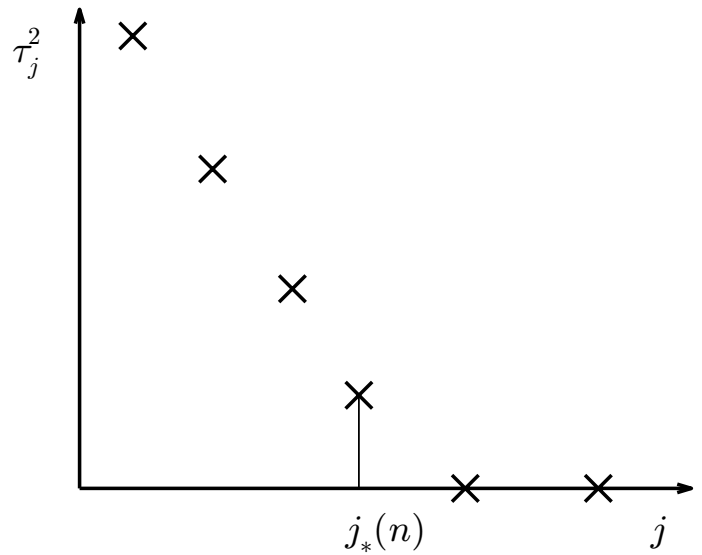
$$\{f \ : \ \int_0^1 (D^\alpha f)^2 \ \leq C^2 \quad \}$$

Least favourable prior on wavelet coeffs (for sample size n):

$$\theta_{jk} \overset{indep}{\sim} N(0, \tau_j^2)$$

$$\tau_j^2 = \sigma_n^2(\lambda_n 2^{-j\alpha} - 1)_+$$

$$\lambda_n = c(C/\sigma_n)^{2\alpha/(2\alpha+1)}$$



$\Rightarrow$ critical level $j_* = j_*(n, \alpha)$ grows with n

(**Growing Gaussian model** again)

Validity of B-vM depends on level

Bayes estimator for the Pinsker prior attains exact minimax MSE (asymptotically)
But Bernstein von Mises fails at the critical level $j_*(n)$:

At $j_*(n)$ $\quad \tau_{j_*}^2/\sigma_n^2 \le 2^\alpha - 1$
[fine scale features]

$$1 - w_n = \frac{1}{1+\tau_{j_*}^2/\sigma_n^2} \ge 2^{-\alpha} \quad \textbf{BvM fails}$$

At fixed j_0**:** $\quad p = 2^{j_0}$ FIXED (or slowly $\nearrow$)
[coarse scale] $\quad \tau_{j_0}^2/\sigma_n^2 = \lambda_n 2^{-j_0\alpha} \to \infty$

$$1 - w_n \to 0 \qquad \textbf{BvM holds}$$

$\Rightarrow$ Difficulty lies with high dimensional features.

Three Talks

1. **Function Estimation & Classical Normal Theory**
   - $X_n \sim N_{p(n)}(\theta_n, I)$ $\quad$ $p(n) \nearrow$ with n $\quad (MVN)$

2. **The Threshold Selection Problem**
   - In (MVN) with, say, $\hat{\theta}_i = X_i I\{|X_i| > \hat{t}\}$
   - How to select $\hat{t} = \hat{t}(X)$ "reliably"?

3. **Large Covariance Matrices**
   - $X_n \sim N_{p(n)}(I \otimes \Sigma_{p(n)})$; especially $X_n = \begin{bmatrix} Y_n \\ Z_n \end{bmatrix}$
   - spectral properties of $n^{-1} X_n X_n^T$
   - PCA, CCA, MANOVA